Fractal Based Diffusion Modeling and Development of Physical Reservoir Computing Performance Metrics

ABSTRACT

This study first investigates and develops models for fractal based structures. Fractal structures are characterized by the self-similarity they exhibit at different length and time scales. This self-similarity can be characterized using a measure called the fractal dimension, which quantifies the complexity by the ratio of the change in geometry by the change in scale. This measure is used to describe these geometries mathematically by employing fractal order differential operators related to the structure's fractal dimension. Fractal structures arise remarkably frequently in nature; some examples include snowflakes, lightning and blood vessels. Complex multi-scale material structures exhibit the same self-similar properties of fractals and therefore it is useful to model these structures in a fractal mathematical framework to better understand their behavior. The first fractal analyzed in this work is built using Diffusion-limited aggregation (DLA), which is an iterative process that produces a two-dimensional fractal structure, whereby particles undergo a random walk and cluster together to form larger structures of particles. The structures produced using this process can be observed in many natural systems such as electrodeposition and dielectric breakdown. A finite difference model is developed to simulate heat diffusion on different DLA structures with varying fractal dimension. The effect of the structure's fractal dimension on the rate of heat transfer across the structure is analyzed.

Expanding on insights analyzing two-dimensional fractal structures, heat transport through 3D printed fractal media is investigated by comparing a fractal diffusion model to infrared measurements using Bayesian uncertainty quantification. The Delayed Rejection Adaptive Metropolis (DRAM) algorithm, based on the Markov Chain Monte Carlo (MCMC) sampling technique, is used to infer parameter uncertainty, quantify parameter correlation, and compute error propagation of the temperature distributions. The results demonstrate that fractal operators improve modeling thermal transport through complex fractal structures and help understand fractal structure-property relationships. For example, correlations among fractal spatial and temporal scaling parameters, diffusion coefficients, and fractal dimensions are quantified. This work finds a scaling relationship between the diffusion coefficient D and the temporal fractal time derivative order α that scales nominally as $D \propto e^{-\alpha}$ based on constraints from the second law of thermodynamics. The results have implications for building a stronger understanding of heat transport in complex materials beyond random media and models based on Gaussian probability homogenization.

Building on this fractal analysis, a multifractal approach is used to explore these complex structures further. Multifractal analysis originated from the desire to obtain a deeper understanding of materials that follow random fractal structures. Complex systems display diverse degrees of irregularity and self-similarity on multiple scales that cannot be adequately described by a single fractal dimension due to uncertainties as well as deterministic changes in structure at different length and time scales. The multifractal spectrum measures the distribution

of a specific property across different scales or levels within a system using a range of exponents. These properties can encompass anything from turbulence intensity in fluid flow to the distribution of heat flow through a solid. Each exponent in the spectrum corresponds to a particular level of irregularity, and a broad range of exponents indicates more complexity. Multifractal measures have a range of applications, from biology to mechanical engineering, among many others. In multifunctional materials used in adaptive systems, a more accurate description of material properties, such as viscoelasticity, heat transport, and phase transformations, often involves considering information about their underlying fractal material structure. This study goes beyond fractals to include randomness in terms of multifractal measures. This involves analysis of the Renyi entropy. This study evaluates this approach by examining heat transport in DLA structures which display multifractal properties. Renyi entropy is relevant here due to its entropy order parameter's close connection to the multifractal spectrum. In fact, the multifractal spectrum can be directly calculated through the generalized Renyi entropy dimension and a box-counting process. This work investigates the multifractality of different DLA structures and leverages this insight to better understand heat diffusion processes using fractal-order diffusion equations; however, the methodology is more broadly applicable to a wide range of multiphysics problems involving energy transport over complex media.

This study goes on to apply the insights and methods developed in the multifractal analysis to physical reservoir computer structures. Physical reservoir computers combine sensing physical inputs and information processing in one single component to reduce the delay caused by data transmission. Physical reservoir computing uses the nonlinear dynamics of physical systems to process information and acts as a neural network for machine learning, allowing for prediction of target signals in real-time. This method significantly reduces the amount of computing and time required for the control system to respond to stimuli. The geometry and physical properties of the physical reservoir structure govern the overall performance. The structure must have enough complexity and nonlinearity when stimulated while not descending into unpredictable chaos to maximize its performance. This work investigates the utilization of fractal structures in the physical reservoir computing framework, and aims to tune the physical reservoir computer using the multifractal spectrum to find the optimal range that maximizes the information processing potential of the system.

Physical reservoir computing (PRC) performance is known to strongly depend on nonlinear properties. This complicates identifying appropriate metrics that distinguish one PRC from another. This work evaluates how nonlinear elasticity influences a PRC's performance in tracking a variety of target signals. Different information theoretic measures ranging from Renyi entropy, multifractal spectra, mutual information and Fisher information are applied to understand the PRC's ability to track a target signal based upon different reference signals. The information processing ability of a mechanical PRC is first elucidated by introducing a simple three node causal structure (directed acyclic graph, DAG) that fuses information between discrete sensors and the PRC's dynamic states. The three node DAG is motivated by the Monty Hall problem, where the PRC serves as the new information similar to increasing odds of winning the car hidden behind one of three doors in the game show Let's Make a Deal. This concept is extended to a continuous system, rather than three doors, to model continuously varying nonlinear elastic structures. In addition, the PRC is assumed to exhibit power law behavior between the input and output. The data processed through the PRC is combined with linear sensor measurements. This work illustrates how mutual information and Fisher information lead to increased knowledge of the system as a function of the power law nonlinearity in the PRC. The three node DAG is then extended to a higher fidelity dynamical nonlinear model to assess PRC performance further. Conventional triple sinusoidal input signals and nonlinear autoregressive moving averages (NARMA) are compared to other input and output signals that contain varying levels of complexity. This work shows how each information theoretic metric provides new insights on predicting ``the edge of chaos'' (often associated with optimal performance) before training a PRC on a target signal. Multifractal spectra are shown to be key to understanding ``the edge of chaos''. It is known from biological systems that increases in the width of the multifractal spectrum lead to rich dynamics that can adapt to different stimuli. This study observes similar characteristics in the dynamic states of the nonlinear mechanical PRCs. The resulting correlation is evaluated using Renyi entropy given the direct connection between multifractal spectra and Renyi entropy. This connection provides a deeper relationship that can be used to facilitate the design of a PRC by quantifying the complexity of the system's expected input signals and the desired target signals. For practical aerodynamic state estimation applications, this study takes pressure measurements from experimental supersonic cavity flow as the input to the PRC. It is computationally shown that mutual information provides a good measure to design nonlinearities into a PRC.

Lastly, this work develops training methods based on wind tunnel experimental data for PRC structures that allow for the ability to influence flow phenomena to cover a wide regime of flow conditions at the PRC surface for PRC training. To accomplish this, a piezoelectric vortex generator (VG) structure is employed to influence flow downstream. The VG fins are actuated in the direction normal to the aerodynamic surface by a piezoelectric actuator, allowing for the height of the VG fins to be controlled. To analyze how actuation of the VG height propagates to the flow phenomena, the multifractal spectra is employed. This provides a measure of multiscale complexities in spatial or temporal data which is important to understanding information processing in PRCs. The shape and width of the multifractal spectrum may vary depending on these multiscale complexities within the dataset. The relationships between the multifractal spectrum of the VG piezoelectric actuator input and the flow phenomena at the aerodynamic surface of the PRC input are analyzed to provide a deeper understanding to design PRC structures on an aerodynamic surface such that the information transfer from input flow dynamics to the PRC input is maximized. The piezoelectric actuated VG also allows for a diverse training regime with a range of different flow conditions, and thus improving PRC training and potentially expanding the sensing capability to a wider set of flow conditions.